

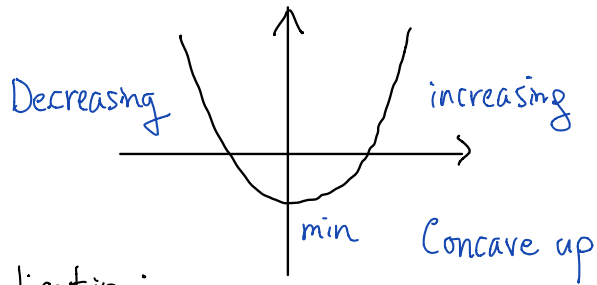
# Math 2010 Advanced Calculus I

## Differential Calculus of Multi-variable Functions

For one-variable function:

We study  $f(x)$ , eg.  $f(x) = x^2 - 1$

Differentiation:  $f'(x) = 2x$ ,  $f''(x) = 2$



Application:

- Curve Sketching
- Rate of Change: velocity, acceleration...
- Finding max/min
- Approximation

How about vector-valued multi-variable functions?

Objects of studies:  $\vec{f}(\vec{x}) = \vec{f}(x_1, x_2, \dots, x_n)$

eg

multi-variable      vector-valued

$$f: \mathbb{R}^3 \rightarrow \mathbb{R}$$

$$g: \overbrace{\mathbb{R}^2}^{\text{multi-variable}} \rightarrow \underbrace{\mathbb{R}^2}_{\text{vector-valued}}$$

$$f(x, y, z) = x \sin y \cos z \quad g(x, y) = (x^2 + y, e^{xy})$$

Want to do similar things:

- Study graph of functions
- Rate of Change with respect to each variable or any direction
- Finding max/min
- Approximation

# Euclidean Space $\mathbb{R}^n$ (An.1.1, Thomas 12.1-12.3)

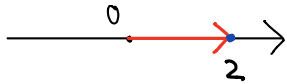
$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R} \quad (n \text{ copies of } \mathbb{R})$$

$$= \{(x_1, x_2, \dots, x_n) : x_i \in \mathbb{R} \text{ for } 1 \leq i \leq n\}$$

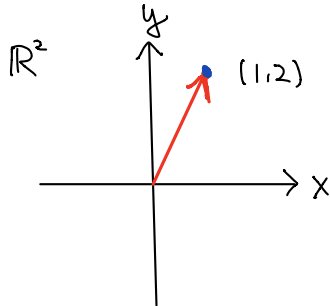
Cartesian/Rectangular coordinates

## Pictures

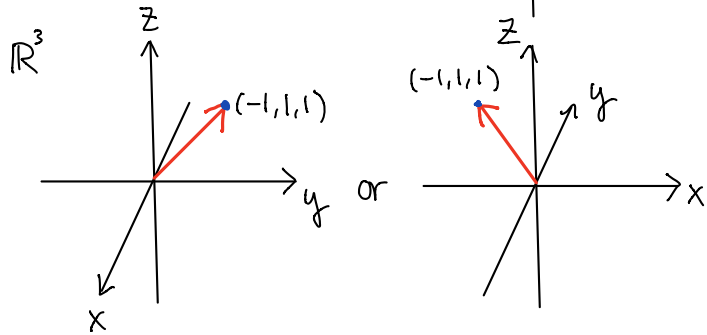
$\mathbb{R}^1$



$\mathbb{R}^2$



$\mathbb{R}^3$



$\mathbb{R}^4$  or higher dimension: Difficult to draw

## Rmk

Each  $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  can be viewed as a point or a vector in  $\mathbb{R}^n$ .

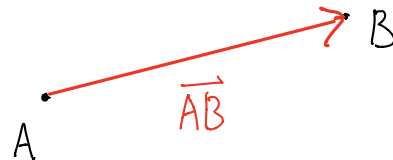
Notations:

1.  $\vec{X} = \mathbf{X} = X = (x_1, x_2, \dots, x_n)$

2.  $\vec{AB}$  is a vector with

initial point A, terminal point B

length/magnitude  $|\vec{AB}|$  or  $\|\vec{AB}\|$



## Basic operations of vectors

$$\text{Let } \vec{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$$

$$\vec{b} = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$$

$$r \in \mathbb{R}$$

### Equality

$$\vec{a} = \vec{b} \Leftrightarrow a_i = b_i \text{ for } i=1, 2, \dots, n$$

### Addition

$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

### Scalar multiplication

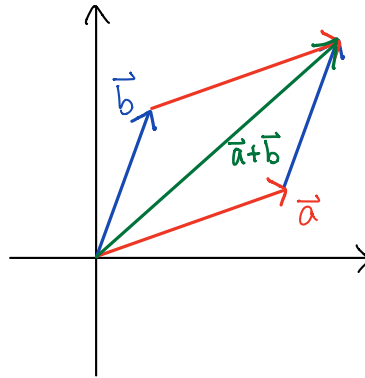
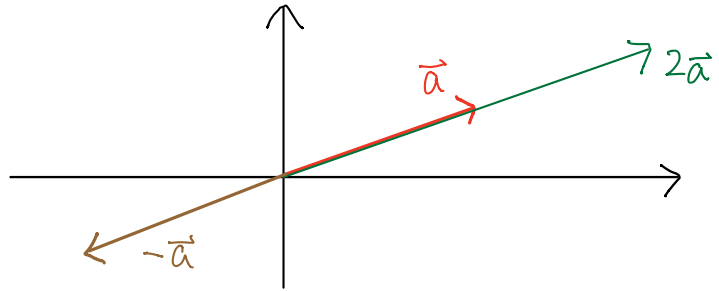
$$r\vec{a} = (ra_1, ra_2, \dots, ra_n)$$

### Subtraction

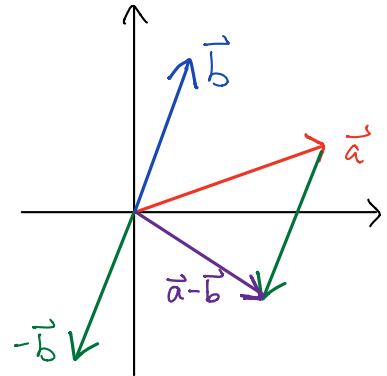
$$\vec{a} - \vec{b} = \vec{a} + (-1)\vec{b}$$

$$= (a_1 - b_1, a_2 - b_2, \dots, a_n - b_n)$$

Pictures     $\vec{a} = (3, 1)$      $\vec{b} = (1, 2)$



Parallelogram law



## Length, Dot Product in $\mathbb{R}^n$

### Defn

$$\text{Let } \vec{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$$

$$\vec{b} = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$$

Define dot product

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

Warning: Don't forget the dot!

In particular,

$$\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + \dots + a_n^2$$

$$= \|\vec{a}\|^2$$

Another notation

$$\langle \vec{a}, \vec{b} \rangle = \vec{a} \cdot \vec{b}$$

(Standard) inner product

## Properties of Dot Product

Let  $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^n$ ,  $r \in \mathbb{R}$ . Then

$$\textcircled{1} (\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$$

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

$$\textcircled{2} (r\vec{a}) \cdot \vec{b} = a \cdot (r\vec{b}) = r(\vec{a} \cdot \vec{b})$$

$$\textcircled{3} \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

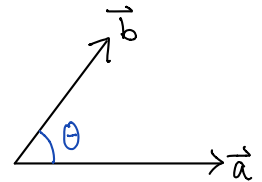
$$\textcircled{4} \vec{a} \cdot \vec{a} \geq 0$$

with equality holds  $\Leftrightarrow \vec{a} = \vec{0} = (0, 0, 0, \dots, 0)$

zero vector

$$\textcircled{5} \vec{a} \cdot \vec{a} = \|\vec{a}\|^2$$

$$\textcircled{6} \vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta, \text{ where } \theta \text{ is the angle between } \vec{a} \text{ and } \vec{b}$$



Hence, if  $\vec{a}, \vec{b} \neq \vec{0}$ ,

$$\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \cos \theta = 0 \Leftrightarrow \vec{a} \perp \vec{b}$$

Rmk

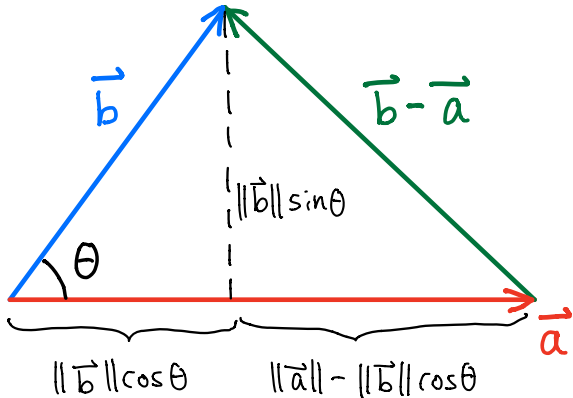
For  $n \leq 3$ , (5), (6) are geometric properties

↓ Generalize

For  $n \geq 4$ , (5), (6) are definitions  
for  $\|\vec{a}\|$  and  $\theta$

Pf of (6) (for  $n \leq 3$ )

Assume  $\theta \leq \frac{\pi}{2}$



Note

$$\begin{aligned}\|\vec{b} - \vec{a}\|^2 &= (\vec{b} - \vec{a}) \cdot (\vec{b} - \vec{a}) \\ &= \vec{b} \cdot \vec{b} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{a} \cdot \vec{a} \\ &= \|\vec{b}\|^2 + \|\vec{a}\|^2 - 2\vec{a} \cdot \vec{b} \quad \textcircled{I}\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\vec{b} - \vec{a}\|^2 &= (\|\vec{a}\| - \|\vec{b}\| \cos \theta)^2 + (\|\vec{b}\| \sin \theta)^2 \\ &= \|\vec{a}\|^2 - 2\|\vec{a}\| \|\vec{b}\| \cos \theta + \|\vec{b}\|^2 \cos^2 \theta \\ &\quad + \|\vec{b}\|^2 \sin^2 \theta \\ &= \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\| \|\vec{b}\| \cos \theta \quad \textcircled{II}\end{aligned}$$

Compare  $\textcircled{I}$  and  $\textcircled{II} \Rightarrow \textcircled{6}$

Ex Try the case  $\theta > \frac{\pi}{2}$

eg Let  $\vec{v}, \vec{w}$  have same length

Show that  $(\vec{v} + \vec{w}) \cdot (\vec{v} - \vec{w}) = 0$

Sol

$$\text{L.H.S.} = (\vec{v} + \vec{w}) \cdot (\vec{v} - \vec{w})$$

$$= \vec{v} \cdot \vec{v} - \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{v} - \vec{w} \cdot \vec{w}$$

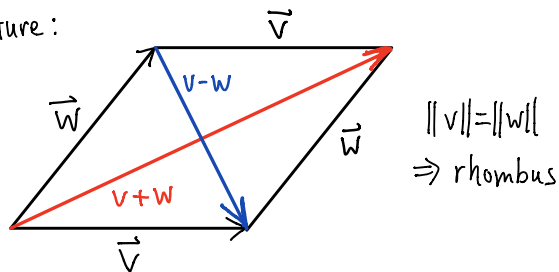
$$= \|\vec{v}\|^2 - \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{w} - \|\vec{w}\|^2$$

$$= \|\vec{v}\|^2 - \|\vec{w}\|^2 \quad (\text{same length})$$

$$= 0$$

$$\Rightarrow (\vec{v} + \vec{w}) \perp (\vec{v} - \vec{w})$$

Picture:

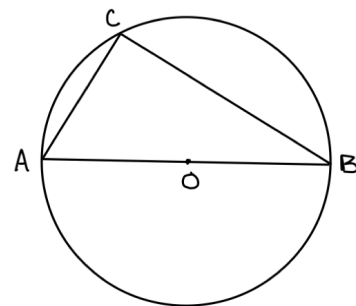


Geometric meaning

Diagonals of a rhombus are perpendicular

eg Consider a circle centered at  $O$ .

$AB$  is diameter.



Show that  $\angle ACB = 90^\circ$

Sol

$$\vec{AC} = \vec{AO} + \vec{OC}$$

$$\vec{BC} = \vec{BO} + \vec{OC} = -\vec{AO} + \vec{OC}$$

$$\vec{AC} \cdot \vec{BC} = (\vec{AO} + \vec{OC}) \cdot (-\vec{AO} + \vec{OC})$$

$$= -\vec{AO} \cdot \vec{AO} + \vec{AO} \cdot \vec{OC} - \vec{OC} \cdot \vec{AO} + \vec{OC} \cdot \vec{OC}$$

$$= -\|\vec{AO}\|^2 + \|\vec{OC}\|^2 \quad \left( \begin{array}{l} \|\vec{AO}\| = \|\vec{OC}\| \\ \text{are radius} \end{array} \right)$$

$$= 0$$

$$\therefore \vec{AC} \perp \vec{BC} \Rightarrow \angle ACB = 90^\circ$$

## Two inequalities (Au 1.2)

### Cauchy-Schwarz Inequality

Let  $\vec{a}, \vec{b} \in \mathbb{R}^n$ . Then

$$|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \|\vec{b}\|$$

i.e.  $\left| \sum_{i=1}^n a_i b_i \right| \leq \sqrt{\sum_{i=1}^n a_i^2} \cdot \sqrt{\sum_{i=1}^n b_i^2}$

Equality holds  $\iff \vec{a} = r\vec{b}$  or  $\vec{b} = r\vec{a}$   
for some  $r \in \mathbb{R}$

$$|\vec{a} \cdot \vec{b}| = \|\vec{a}\| \|\vec{b}\|$$

Rmk For  $n \leq 3$ ,

Cauchy-Schwarz inequality follows from

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$$

Pf

Case 1: If  $\vec{a} = \vec{0}$  or  $\vec{b} = \vec{0}$

$$\text{L.H.S.} = \text{R.H.S.} = 0$$

Equality holds and  $\vec{a} = 0\vec{b}$  or  $\vec{b} = 0\vec{a}$

Case 2:  $\vec{a}, \vec{b} \neq \vec{0}$

Let  $f(t) = \|t\vec{a} - \vec{b}\|^2$  for  $t \in \mathbb{R}$

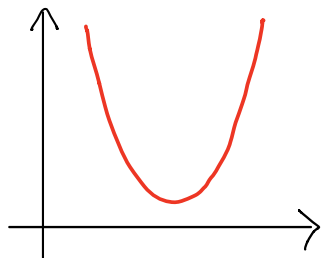
Then

$$\begin{aligned} f(t) &= (t\vec{a} - \vec{b}) \cdot (t\vec{a} - \vec{b}) \\ &= t^2 \vec{a} \cdot \vec{a} - t \vec{a} \cdot \vec{b} - t \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} \end{aligned}$$

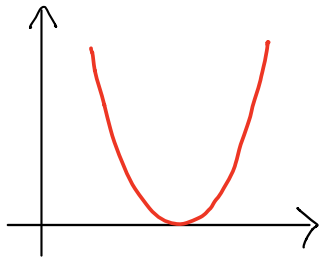
Note  $f(t) = \|t\vec{a} - \vec{b}\|^2 \geq 0$  for any  $t$

Note that

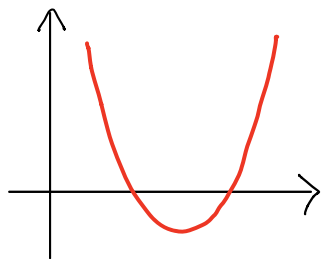
- ①  $f(t)$  is a real quadratic polynomial with leading coefficient  $\|\vec{a}\|^2 > 0$
- ② For a real quadratic polynomial with positive leading coefficient



discriminant  $\Delta < 0$   
(no real root)



$\Delta = 0$   
(repeated root)



$\Delta > 0$   
(distinct real roots)

$$f(t) = \|\vec{a}\|^2 t^2 - 2\vec{a} \cdot \vec{b} t + \|\vec{b}\|^2 \geq 0 \quad \forall t \in \mathbb{R}$$

$$\Rightarrow \Delta \leq 0$$

$$\Rightarrow (-2\vec{a} \cdot \vec{b})^2 - 4(\|\vec{a}\|^2)(\|\vec{b}\|^2) \leq 0$$

$$\Rightarrow (\vec{a} \cdot \vec{b})^2 \leq \|\vec{a}\|^2 \|\vec{b}\|^2$$

$$\Rightarrow |\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \|\vec{b}\|$$

Also,

$$\text{Equality holds} \Leftrightarrow \Delta = 0$$

$$\Leftrightarrow f(t) = \|t\vec{a} - \vec{b}\|^2 \text{ has a repeated root}$$

$$\Leftrightarrow f(r) = \|r\vec{a} - \vec{b}\|^2 = 0 \text{ for some } r \in \mathbb{R}$$

$$\Leftrightarrow r\vec{a} - \vec{b} = \vec{0} \text{ for some } r \in \mathbb{R}$$

$$\Leftrightarrow \vec{b} = r\vec{a} \text{ for some } r \in \mathbb{R}$$



Rmk Let  $\vec{a}, \vec{b} \neq \vec{0} \in \mathbb{R}^n$

We showed that if  $n \leq 3$ ,

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$$

$$\Rightarrow \theta = \cos^{-1} \left( \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} \right)$$

For any  $n \geq 1$

Cauchy-Schwarz inequality

$$\Rightarrow -1 \leq \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} \leq 1$$

Hence, we can define

$$\theta = \cos^{-1} \left( \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} \right) \in [0, \pi]$$

to be the angle between

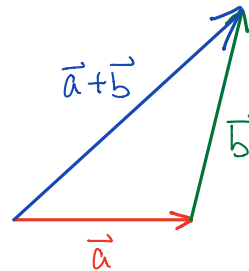
$\vec{a}$  and  $\vec{b}$  for  $n \geq 4$

### Triangle Inequality

Let  $\vec{a}, \vec{b} \in \mathbb{R}^n$ . Then

$$\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$

Equality holds  $\Leftrightarrow \vec{a} = r\vec{b}$  or  $\vec{b} = r\vec{a}$   
for some  $r \geq 0$



Pf  $\|\vec{a} + \vec{b}\|^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$

$$= \vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}$$

$$\leq \|\vec{a}\|^2 + 2|\vec{a} \cdot \vec{b}| + \|\vec{b}\|^2$$

$$\leq \|\vec{a}\|^2 + 2\|\vec{a}\| \|\vec{b}\| + \|\vec{b}\|^2 \quad (\text{Cauchy-Schwarz})$$

$$= (\|\vec{a}\| + \|\vec{b}\|)^2$$

$$\Rightarrow \|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$

$$\text{Equality holds} \Leftrightarrow \|\vec{a}\| \|\vec{b}\| \stackrel{\text{I}}{=} |\vec{a} \cdot \vec{b}| \Leftrightarrow \underbrace{\vec{a} = r\vec{b} \text{ or } \vec{b} = r\vec{a}}_{\text{with } r \geq 0} \stackrel{\text{II}}{=} \vec{a} \cdot \vec{b}$$

Rmk Cauchy-Schwarz inequality  $\Leftrightarrow$  Triangle inequality

( $\Rightarrow$ : Pf above,  $\Leftarrow$  Ex)

## Cross Product (defined only in $\mathbb{R}^3$ ) (Au 1.2 Thomas 12.4)

Let  $\vec{a} = (a_1, a_2, a_3)$ ,  $\vec{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$

Define their cross product to be

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \left( \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)\end{aligned}$$

where  $\hat{i} = (1, 0, 0)$ ,  $\hat{j} = (0, 1, 0)$ ,  $\hat{k} = (0, 0, 1)$

Recall:  $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

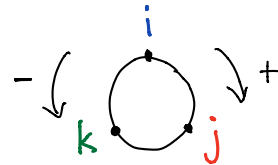
$$\begin{aligned}\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\end{aligned}$$

eg Let  $\vec{a} = (2, 3, 5)$ ,  $\vec{b} = (1, 2, 3)$

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix} \hat{i} - \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} \hat{j} + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \hat{k} \\ &= -\hat{i} - \hat{j} + \hat{k} \\ &= (-1, -1, 1)\end{aligned}$$

Remark

$i \times i = \vec{0}$	$i \times j = k$	$i \times k = -j$
$j \times i = -k$	$j \times j = \vec{0}$	$j \times k = i$
$k \times i = j$	$k \times j = -i$	$k \times k = \vec{0}$



## Properties of Cross Product

Let  $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ ,  $\alpha, \beta \in \mathbb{R}$

Algebraic (follow from property of determinant)

$$\textcircled{1} \quad \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$\textcircled{2} \quad (\alpha \vec{a} + \beta \vec{b}) \times \vec{c} = \alpha \vec{a} \times \vec{c} + \beta \vec{b} \times \vec{c}$$

$$\vec{a} \times (\beta \vec{b} + \gamma \vec{c}) = \beta \vec{a} \times \vec{b} + \gamma \vec{a} \times \vec{c}$$

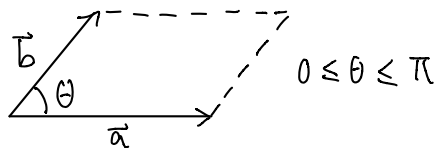
Note:  $\textcircled{1} \Rightarrow \vec{a} \times \vec{a} = -\vec{a} \times \vec{a} \Rightarrow \vec{a} \times \vec{a} = \vec{0}$

## Geometric

$\textcircled{3}$  Let  $\theta$  be the angle between  $\vec{a}, \vec{b}$

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta$$

= Area of parallelogram spanned by  $\vec{a}, \vec{b}$



$$\therefore \vec{a} \times \vec{b} = \vec{0}$$

$$\Leftrightarrow \text{Area} \left( \begin{array}{c} \vec{b} \\ \vec{a} \end{array} \right) = 0$$

$$\Leftrightarrow \vec{a} = r\vec{b} \text{ or } \vec{b} = r\vec{a} \text{ for some } r \in \mathbb{R}$$

$$\Leftrightarrow \{\vec{a}, \vec{b}\} \text{ is linearly dependent}$$

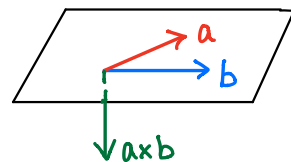
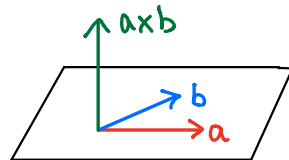
(see linear algebra)

$$\textcircled{4} \quad (\vec{a} \times \vec{b}) \cdot \vec{a} = (\vec{a} \times \vec{b}) \cdot \vec{b} = 0 \quad (\text{Exercise})$$

If  $\vec{a} \times \vec{b}$  is non-zero, then

$$\Rightarrow \vec{a} \times \vec{b} \perp \vec{a}, \quad \vec{a} \times \vec{b} \perp \vec{b}$$

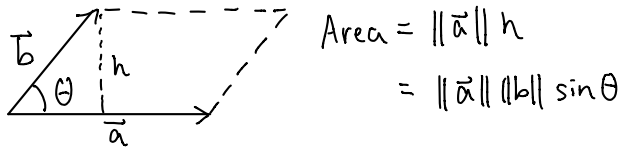
Also,  $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$  satisfy right hand rule



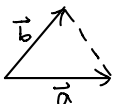
### Pf of ③

By straight forward expansion

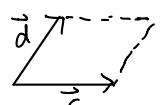
$$\begin{aligned}\|\vec{a} \times \vec{b}\|^2 &= \|\vec{a}\|^2 \|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2 \\ &= \|\vec{a}\|^2 \|\vec{b}\|^2 (1 - \cos^2 \theta) \\ &= \|\vec{a}\|^2 \|\vec{b}\|^2 \sin^2 \theta\end{aligned}$$



### Rmk

• Area of  =  $\frac{1}{2} \|\vec{a} \times \vec{b}\|$

• If  $\vec{c}, \vec{d} \in \mathbb{R}^2$ , then

Area of  =  $\left| \det \begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix} \right|$

### Triple Product (only in $\mathbb{R}^3$ )

Let  $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ . order matters

Their triple product of  $\vec{a}, \vec{b}, \vec{c}$  is defined to be

$$\begin{aligned}(\vec{a} \times \vec{b}) \cdot \vec{c} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot (c_1, c_2, c_3) \\ &= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = - \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\end{aligned}$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

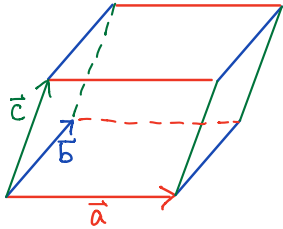
\*

Rmk It follows from (\*) that

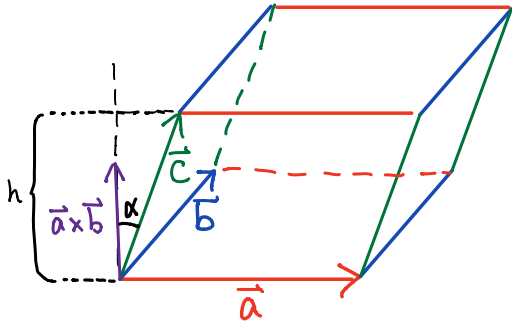
$$\begin{aligned}(\vec{a} \times \vec{b}) \cdot \vec{c} &= (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b} \\ &= -(\vec{b} \times \vec{a}) \cdot \vec{c} = -(\vec{c} \times \vec{b}) \cdot \vec{a} = -(\vec{a} \times \vec{c}) \cdot \vec{b}\end{aligned}$$

## Geometric Meaning

$$|(\vec{a} \times \vec{b}) \cdot \vec{c}| = \text{Volume of parallelepiped spanned by } \vec{a}, \vec{b}, \vec{c}$$



Pf

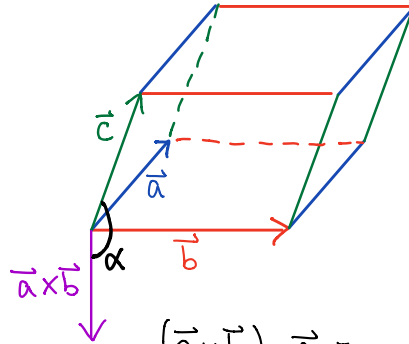


$$\left(\alpha \leq \frac{\pi}{2}\right)$$

For  $\alpha \leq \frac{\pi}{2}$ ,

$$\begin{aligned}(\vec{a} \times \vec{b}) \cdot \vec{c} &= \|\vec{a} \times \vec{b}\| \|\vec{c}\| \cos \alpha \\ &= \|\vec{a} \times \vec{b}\| h \\ &= \text{Base Area} \times \text{height} \\ &= \text{Volume of parallelepiped}\end{aligned}$$

The case for  $\frac{\pi}{2} < \alpha \leq \pi$  is similar

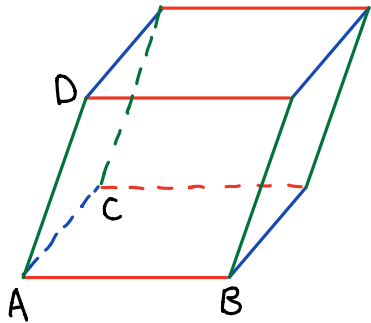


$$(\vec{a} \times \vec{b}) \cdot \vec{c} = - \text{Volume of parallelepiped}$$

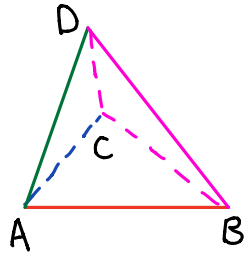
Rmk

$$\begin{aligned}(\vec{a} \times \vec{b}) \cdot \vec{c} &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 \Leftrightarrow \text{Volume of parallelepiped} = 0 \\ &\Leftrightarrow \{\vec{a}, \vec{b}, \vec{c}\} \text{ is lin dependent}\end{aligned}$$

Given 4 points  $A, B, C, D$  in  $\mathbb{R}^3$



Parallelepiped



Tetrahedron

Volume of Tetrahedron

$$= \frac{1}{3} \text{Area}(\triangle ABC) \cdot \text{height}$$

$$= \frac{1}{3} \cdot \frac{1}{2} \text{Area} \left( \begin{array}{c} C \\ \text{parallelogram} \\ A \quad B \end{array} \right) \cdot \text{height}$$

$$= \frac{1}{6} \text{Volume of Parallelepiped}$$

eg Let  $A = (1, 0, 1)$   $B = (1, 1, 2)$   
 $C = (2, 1, 1)$   $D = (2, 1, 3)$

Find volume of tetrahedron ABCD

Sol

$$\vec{AB} = (1, 1, 2) - (1, 0, 1) = (0, 1, 1)$$

$$\vec{AC} = (1, 1, 0)$$

$$\vec{AD} = (1, 1, 2)$$

$$(\vec{AB} \times \vec{AC}) \cdot \vec{AD} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = -2$$

$$\Rightarrow \text{volume of tetrahedron} = \frac{1}{6} |-2| = \frac{1}{3}$$